

Power monoids and their arithmetic

Andreas Reinhart

University of Graz
Department of Mathematics and Scientific Computing

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Monoids, units and atoms

In this talk a monoid is always a commutative semigroup with identity.

Let M be an additively written monoid. Set

$$M^\times = \{x \in M \mid x + y = 0 \text{ for some } y \in M\},$$

called the **unit group** of M .

Set $\mathcal{A}(M) =$

$$\{x \in M \setminus M^\times \mid \text{for all } y, z \in M, x = y + z \Rightarrow y \in M^\times \text{ or } z \in M^\times\},$$

called the **set of atoms** of M .

Length sets

For each $a \in M$, let

$$L(a) = \{n \in \mathbb{N}_0 \mid a = \sum_{i=1}^n u_i \text{ for some atoms } (u_i)_{i=1}^n \text{ of } M\},$$

called the **length set** of a .

Let

$$\mathcal{L}(M) = \{L(a) \mid a \in M, L(a) \neq \emptyset\},$$

called the **system of length sets** of M .

Elasticity and atomicity

For each $a \in M$, let

$$\rho(a) = \begin{cases} \sup(L(a)) / \min(L(a)) & \text{if } \min(L(a)) > 0 \\ 1 & \text{else} \end{cases},$$

called the **elasticity** of a .

Set

$$\rho(M) = \sup\{\rho(a) \mid a \in M\},$$

called the **elasticity** of M .

We say that M is **atomic** if every nonunit of M is a finite sum of atoms.

Reduced monoids and the factorization monoid

We say that M is **reduced** if $M^\times = \{0\}$.

If M is reduced, $Z(M)$ is the free abelian monoid with basis $\mathcal{A}(M)$.

$Z(M)$ is called the **factorization monoid** and it is written multiplicatively.

Let $\pi : Z(M) \rightarrow M$ be the **factorization homomorphism** defined by $\pi(\prod_{i=1}^n a_i) = \sum_{i=1}^n a_i$ for all $n \in \mathbb{N}_0$ and atoms $(a_i)_{i=1}^n$ of M .

For each $a \in M$, set

$$Z(a) = \{z \in Z(M) \mid \pi(z) = a\},$$

called the **set of factorizations** of a .

Relatively prime and (relatively) cancellative elements

If $a, b \in M$, then we say that a and b are **relatively prime** if for all $c, d, e \in M$ with $a = c + d$ and $b = c + e$, it follows that $c \in M^\times$.

Let $a \in M$.

We say that a is **cancellative** if for all $b, c \in M$ with $a + b = a + c$, it follows that $b = c$.

Moreover, a is called **relatively cancellative** if for all $b, c, d \in M$ with $a = b + c = b + d$, it follows that $c = d$.

We say that M is **cancellative** if every element of M is cancellative.

Moreover, M is called **unit cancellative** if for all $b, c \in M$ with $b = b + c$, it follows that $c \in M^\times$.

On the cancellativity conditions

- Units are cancellative and cancellative elements are relatively cancellative.
- If M is reduced, then each atom of M is relatively cancellative.
- Divisors of (relatively) cancellative elements are (relatively) cancellative.
- Sums of cancellative elements are cancellative.
- Sums of relatively cancellative elements need not be relatively cancellative.

Let R be an integral domain and \mathcal{I} be the monoid of nonzero ideals of R .

- Every cancellative monoid is unit cancellative.
- If R is noetherian, then \mathcal{I} is unit cancellative.
- \mathcal{I} is cancellative if and only if R is an almost Dedekind domain.

BF-/half-factorial/fully elastic monoids

Let M be an atomic monoid. Then M is called a **BF-monoid** if $L(a)$ is finite for each $a \in M$.

We say that M is **half-factorial** if $L(a)$ is a singleton for each $a \in M$.

Furthermore, M is said to be **fully elastic** if for each rational number $1 \leq r < \rho(M)$, there is some $a \in M$ such that $\rho(a) = r$.

Note that if

$$\mathcal{L}(M) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite, } \min(A) \geq 2\},$$

then M is fully elastic.

Power monoids

Let $\mathcal{P}_{\text{fin}}(M)$ be the set of nonempty finite subsets of M and let $\mathcal{P}_{\text{fin},0}(M)$ be the set of finite subsets of M that contain 0.

Let $+: \mathcal{P}_{\text{fin}}(M) \times \mathcal{P}_{\text{fin}}(M) \rightarrow \mathcal{P}_{\text{fin}}(M)$ be defined by $A + B = \{a + b \mid a \in A, b \in B\}$ for each $A, B \in \mathcal{P}_{\text{fin}}(M)$.

Then $\mathcal{P}_{\text{fin}}(M)$ equipped with $+$ is a monoid, called the **finitary power monoid** of M .

Besides that $\mathcal{P}_{\text{fin},0}(M)$ is a submonoid of $\mathcal{P}_{\text{fin}}(M)$, called the **reduced finitary power monoid** of M .

Some properties of power monoids

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ and let $a, b \in \mathbb{N}$ be such that $a < b$.

- $\mathcal{H}^\times = \{\{0\}\}$ is the set of all cancellative elements of \mathcal{H} .
- $\{\{0, n\} \mid n \in \mathbb{N}\} \subseteq \mathcal{A}(\mathcal{H})$.
- $\{0, a, b\} \in \mathcal{A}(\mathcal{H})$ if and only if $\{0, a, b\}$ is not an arithmetical progression.
- \mathcal{H} is a BF-monoid and unit cancellative.

Let $M = (\mathbb{N}, \cdot)$. Then the set of prime numbers \mathbb{P} is the set of atoms of M and $\lim_{n \rightarrow \infty} \frac{|\{m \in \mathbb{P} \mid m \leq n\}|}{n} = 0$.

For each $n \in \mathbb{N}$, let $\mathcal{A}_n = \{A \in \mathcal{A}(\mathcal{H}) \mid \max(A) \leq n\}$ and let $\mathcal{P}_n = \{A \in \mathcal{H} \mid \max(A) \leq n\}$. Then $\lim_{n \rightarrow \infty} \frac{|\mathcal{A}_n|}{|\mathcal{P}_n|} = 1$.

Importance of power monoids, part I

Let H and D be monoids. It is clear that if H and D are isomorphic, then $\mathcal{P}_{\text{fin}}(H)$ and $\mathcal{P}_{\text{fin}}(D)$ are isomorphic (and $\mathcal{P}_{\text{fin},0}(H)$ and $\mathcal{P}_{\text{fin},0}(D)$ are isomorphic, too).

Power monoids were introduced and studied by Tamura and Shafer in 1967. Under what circumstances is the converse to the aforementioned facts true?

This is a recent research topic and there are many partial results by Bienvenu, Geroldinger, Rago, Tringali and others.

Importance of power monoids, part II

Let H be a monoid and let $\text{Aut}(H)$ be the automorphism group of H (i.e., the set of all monoid automorphism on H under composition). How does $\text{Aut}(H)$ look like?

Theorem (Tringali, Yan, 2025)

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. Then $\text{Aut}(\mathcal{H}) = \{\text{id}, \sigma\}$, where id is the identity map and $\sigma : \mathcal{H} \rightarrow \mathcal{H}$ is defined by $\sigma(X) = \{\max(X) - x \mid x \in X\}$ for each $X \in \mathcal{H}$.

For more results on this topic see, for instance, the papers of Rago, Tringali, Wen.

Importance of power monoids, part III

Let M be an additively written monoid and let $\mathcal{H} \in \{\mathcal{P}_{\text{fin}}(M), \mathcal{P}_{\text{fin},0}(M)\}$. What can be said about the arithmetic of \mathcal{H} ? This includes (but is not limited to) the following problems.

- Characterize when \mathcal{H} is atomic/a BF-monoid/a half-factorial monoid.
- Determine $\rho(\mathcal{H})$, the elasticity of \mathcal{H} .
- Describe $\mathcal{L}(\mathcal{H})$ and characterize when \mathcal{H} is fully elastic.

More recent results were established by Aggarwal, Cossu, Gotti, Lu, Tringali and others.

Motivational results, part I

Theorem (Chapman, McClain, 2005)

Let $D = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the ring of integer-valued polynomials and let $H = D \setminus \{0\}$.

Then H is fully elastic.

Theorem (Frisch, 2013)

Let $D = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the ring of integer-valued polynomials and let $H = D \setminus \{0\}$.

Then $\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite, } \min(A) \geq 2\}$.

Krull monoids

Let H be a (multiplicatively written) cancellative monoid.

$H_{\text{red}} = \{xH^\times \mid x \in H\}$ is the **reduced monoid associated to H** .

Moreover, H is called a **Krull monoid** if we can find a free abelian monoid F such that H_{red} is a submonoid of F such that for all $a, b \in H_{\text{red}}$ and $c \in F$ with $a = bc$, it follows that $c \in H_{\text{red}}$.

Monoids of (plus-minus weighted) zero-sum sequences

Let G be an abelian group, let $G_0 \subseteq G$ and let $\mathcal{F}(G_0)$ be the free abelian monoid with basis G_0 .

Let $\mathcal{B}(G_0) = \{\prod_{i=1}^n g_i \in \mathcal{F}(G_0) \mid \sum_{i=1}^n g_i = 0\}$, called the **monoid of zero-sum sequences** of G_0 (or called the **block monoid** of G_0).

Let

$$\mathcal{B}_{\pm}(G_0) = \left\{ \prod_{i=1}^n g_i \in \mathcal{F}(G_0) \mid \sum_{i=1}^n \varepsilon_i g_i = 0 \text{ for some } \varepsilon_i \in \{-1, 1\} \right\},$$

called the **monoid of plus-minus weighted zero-sum sequences** of G_0 .

Properties of the aforementioned monoids

Let G be an abelian group and let $G_0 \subseteq G$ be a subset.

Then $\mathcal{B}(G_0)$ is a Krull monoid and if H is a Krull monoid, then there exists an abelian group G and a subset $G_0 \subseteq G$ such that H_{red} is isomorphic to $\mathcal{B}(G_0)$.

Also note that $\mathcal{B}_{\pm}(G)$ is a Krull monoid if and only if G is an elementary 2-group.

Motivational results, part II

Theorem (Kainrath, 1999)

Let H be a Krull monoid and let G be an infinite abelian group such that $H_{\text{red}} \cong \mathcal{B}(G)$.

Then $\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite, } \min(A) \geq 2\}$.

Theorem (Geroldinger, Kainrath, 2026)

Let G be an infinite abelian group and let $H = \mathcal{B}_{\pm}(G)$ be the monoid of plus-minus weighted zero-sum sequences of G .

Then $\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite, } \min(A) \geq 2\}$.

A conjecture and some partial results

Conjecture (Fan, Tringali, 2018)

$$\mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)) = \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ finite, } \min(A) \geq 2\}.$$

Theorem (Fan, Tringali, 2018)

$$\{[2, n], \{2, n\}, \{n\} \mid n \in \mathbb{N}_{\geq 2}\} \subseteq \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)).$$

Corollary

$$\mathcal{P}_{\text{fin}}(\mathbb{N}_0) \text{ is fully elastic and } \rho(\mathcal{P}_{\text{fin}}(\mathbb{N}_0)) = \rho(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)) = \infty.$$

Theorem (Antoniou, 2020)

$$\{[2, n] \cup \{n + m\} \mid n, m \in \mathbb{N}_{\geq 2}\} \subseteq \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)).$$

Some complementary results

Theorem (Geroldinger, Halter-Koch, 2006)

Let G be an abelian group and let $G_0 \subseteq G$ be a finite subset. Then $\mathcal{L}(\mathcal{B}(G_0)) \subsetneq \{\{0\}, \{1\}\} \cup \{A \subseteq \mathbb{N}_0 \mid \emptyset \neq A \text{ is finite, } \min(A) \geq 2\}$.
(More is true: In fact $\mathcal{B}(G_0)$ satisfies the structure theorem for sets of lengths.)

Theorem (Baginski, Chapman, Crutchfield, Kennedy, Wright, 2006)

Let K be a field, let $S \neq \mathbb{N}_0$ be a numerical semigroup (i.e., S is an additive submonoid of \mathbb{N}_0 such that $\mathbb{N}_0 \setminus S$ is finite) and let $H = \{f \in K[[X]] \setminus \{0\} \mid f_i = 0 \text{ for all } i \in \mathbb{N}_0 \setminus S\}$. Then H is not fully elastic.

Why relatively cancellative elements are useful

Proposition (R., 2025)

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ and let $X, Y \in \mathcal{H}$ be relatively cancellative such that $\gcd(Y) > 2 \max(X)$.

Then $X + Y$ is relatively cancellative, $Z(X + Y) = Z(X)Z(Y)$ and $L(X + Y) = L(X) + L(Y)$.

Corollary

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ and let $X, Y \in \mathcal{H}$ be relatively cancellative. Then there is some relatively cancellative $Z \in \mathcal{H}$ such that $L(X) + L(Y) = L(Z)$.

Some counterexamples

Examples

Let $A = \{0, 1, 2, 3\}$, $B = \{0, 7\}$, $C = \{0, 1\}$, $D = \{0, 3, 6, 9\}$,
 $E = \{0, 2\}$.

- (1) A and D are not relatively cancellative and B , C and E are relatively cancellative.
- (2) $\gcd(B) > 2 \max(A)$, $\gcd(D) > 2 \max(C)$ and $\gcd(E) = 2 \max(C)$.
- (3) $L(A + B) = \{2, 3, 4\} \not\supseteq \{3, 4\} = L(A) + L(B)$.
- (4) $L(C + D) = \{2, 3, 4\} \not\supseteq \{3, 4\} = L(C) + L(D)$.
- (5) $L(C + E) = \{2, 3\} \not\supseteq \{2\} = L(C) + L(E)$.

How to construct nontrivial relatively cancellative elements

Theorem (R., 2025)

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ and let $(n_j)_{j=0}^\infty$, $(A_j)_{j=0}^\infty$, $(B_j)_{j=0}^\infty$, $(C_j)_{j=0}^\infty$, $(D_j)_{j=0}^\infty$ and $(S_j)_{j=0}^\infty$ be defined recursively as follows.

Set $n_0 = 0$, $A_0 = \{0, 1\}$, $B_0 = \{0\}$,

$C_0 = \{0\}$, $D_0 = \{0, 1\}$ and $S_0 = \{0, 1\}$.

For each $i \in \mathbb{N}_0$, let $n_{i+1} \in \mathbb{N}$ with $n_{i+1} \geq 3 \max(D_i)$,

$A_{i+1} = A_i \cup \{1 + n_{i+1}\}$, $B_{i+1} = B_i \cup (n_{i+1} + D_i)$,

$C_{i+1} = C_i \cup \{n_{i+1}\}$, $D_{i+1} = D_i + \{0, 1 + n_{i+1}\}$ and

$S_{i+1} = C_{i+1} + D_{i+1}$.

Then for each $i \in \mathbb{N}$, $S_i \in \mathcal{H}$ is relatively cancellative,

$Z(S_i) = \{A_i B_i, C_i \prod_{j=0}^i \{0, 1 + n_j\}\}$ and $L(S_i) = \{2, i + 2\}$.

Sketch of the proof I

Step 1: Show by induction that for each $i \in \mathbb{N}$, $A_i, B_i, C_i \in \mathcal{A}(\mathcal{H})$, $D_i \in \mathcal{H}$, $Z(D_i) = \{\prod_{j=0}^i \{0, 1 + n_j\}\}$, $L(D_i) = \{i + 1\}$, $\max(A_i \cup B_i \cup C_i) < \max(D_i)$, $A_i = \{0\} \cup (1 + C_i)$ and $A_i + B_i = C_i + D_i$.

Step 2: Prove that for each $i \in \mathbb{N}$, $Z(S_i) = \{A_i B_i, C_i \prod_{j=0}^i \{0, 1 + n_j\}\}$.

Step 2a: Show that $Z(S_1) = \{A_1 B_1, C_1 \prod_{j=0}^1 \{0, 1 + n_j\}\}$ (straightforward) and observe that $\{A_i B_i, C_i \prod_{j=0}^i \{0, 1 + n_j\}\} \subseteq Z(S_i)$ for each $i \in \mathbb{N}$ by Step 1.

Sketch of the proof II

Step 2b: Let $i \in \mathbb{N}$ be such that

$Z(S_i) \subseteq \{A_i B_i, C_i \prod_{j=0}^i \{0, 1 + n_j\}\}$. Prove that

$Z(S_{i+1}) \subseteq \{A_{i+1} B_{i+1}, C_{i+1} \prod_{j=0}^{i+1} \{0, 1 + n_j\}\}$.

Step 2b(i): Let $z \in Z(S_{i+1})$. Then $z = \prod_{i=1}^{n+1} Y_i$ with $n \in \mathbb{N}$ and $(Y_i)_{i=1}^{n+1}$ atoms of \mathcal{H} . There is some $j \in [1, n+1]$ such that $n_1 \in Y_j$. Without restriction, let $n_1 \in Y_{n+1}$. Set $X = \sum_{i=1}^n Y_i$ and $Y = Y_{n+1}$. Then $S_{i+1} = X + Y$, $C_{i+1} \subseteq Y$ and $X \subseteq D_{i+1}$.

Set $X' = X \cap [0, n_{i+1} - 1]$ and $Y' = Y \cap [0, n_{i+1} - 1]$. Show that $S_i = X' + Y'$.

Conclude that either (1) $X' = A_i$ and $Y' = B_i$ or (2) there is some $W \subseteq [0, i]$ such that $X' = \{\sum_{j \in E} (1 + n_j) \mid E \subseteq W\}$ and $Y' = \{\sum_{j \in E} (1 + n_j) \mid E \subseteq [0, i] \setminus W\} + C_i$.

Sketch of the proof III

Step 2b(ii): First let $X' = A_i$ and $Y' = B_i$.

Prove that $X = A_{i+1}$ and $Y = B_{i+1}$.

Then $z = A_{i+1}B_{i+1}$.

Next let $W \subseteq [0, i]$ be such that $X' = \{\sum_{j \in E} (1 + n_j) \mid E \subseteq W\}$
and $Y' = \{\sum_{j \in E} (1 + n_j) \mid E \subseteq [0, i] \setminus W\} + C_i$.

Show that $W = [0, i]$, $X = \{\sum_{j \in E} (1 + n_j) \mid E \subseteq [0, i + 1]\}$ and
 $Y = C_{i+1}$.

Then $z = C_{i+1} \prod_{j=0}^{i+1} \{0, 1 + n_j\}$.

Step 3: Let $i \in \mathbb{N}$. Conclude that $S_i \in \mathcal{H}$ is relatively cancellative and $L(S_i) = \{2, i + 2\}$. (The former follows from the fact that for all $u, v \in Z(S_i)$ with $\gcd(u, v) \neq 1$, it follows that $u = v$ and the latter is obvious.)

A few consequences

Corollary

Let $\mathcal{U} \subseteq \mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ be the submonoid generated by $\{\{1\}, \{2, n\} \mid n \in \mathbb{N}_{\geq 3}\}$. Then $\mathcal{U} \subseteq \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0))$.

Corollary

If $L \subseteq \mathbb{N}_{\geq 2}$ is a nonempty finite arithmetical progression of length n and $\min(L) \geq 2n$, then $L \in \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0))$.

A complementary result

Proposition (R., 2025)

Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ and let $X \in \mathcal{H}$ and $n \in \mathbb{N}$ be such that $n > 2 \max(X)$.

Set $\mathcal{M} =$

$$\{(A, B, C) \in \mathcal{H}^3 \mid A + B = A + C = X, B \text{ and } C \text{ are rel. prime}\}.$$

Set $\mathcal{N} = \{A \in \mathcal{H} \mid \text{there are } B, C \in \mathcal{H} \text{ such that } (A, B, C) \in \mathcal{M}\}.$

Then $Z(X + \{0, n\}) = \bigcup_{(A,B,C) \in \mathcal{M}} (B \cup (n + C))Z(A)$ and $L(X + \{0, n\}) = 1 + \bigcup_{A \in \mathcal{N}} L(A).$

Corollary

$[k, k + 2] \in \mathcal{L}(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0))$ for each $k \in \mathbb{N}_{\geq 2}.$

Examples I

Example

Let $X = \{0, 1, 2, 4, 5, 6, 10, 11, 12, 14, 15, 16, 20, 21, 22, 24, 25, 26\}$.

$$\begin{aligned} X &= \{0, 1\} + \{0, 1\} + \{0, 4\} + \{0, 10\} + \{0, 10\} \\ &= \{0, 4\} + \{0, 10\} + \{0, 1, 2, 6, 10, 11, 12\} \\ &= \{0, 4\} + \{0, 1, 2, 6, 10, 11, 12, 20, 21, 22\}. \end{aligned}$$

X is not relatively cancellative and $L(X) = \{2, 3, 5\}$.

Examples II

Example

Let $X = \{0, 1, 3, 4, 5, 7, 8, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25\} \cup \{30, 31, 33, 34, 35, 37, 38\}$.

$$\begin{aligned} X &= \{0, 1\} + \{0, 3\} + \{0, 4\} + \{0, 13\} + \{0, 17\} \\ &= \{0, 1, 4\} + \{0, 3, 4\} + \{0, 13\} + \{0, 17\} \\ &= \{0, 4\} + \{0, 1, 3, 4, 13, 14, 16, 18, 20, 21, 30, 31, 33, 34\}. \end{aligned}$$

X is not relatively cancellative and $L(X) = \{2, 4, 5\}$.

Examples III

Example

Let $X = \{0, 1, 4, 5, 10, 11, 12, 14, 15, 16, 19, 20, 21, 22, 25, 26, 29\} \cup \{30, 42, 43, 46, 47, 52, 53, 54, 56, 57, 58, 61, 62, 63, 64, 67, 68, 71, 72\}$.

$$\begin{aligned} X &= \{0, 1\} + \{0, 4\} + \{0, 10\} + \{0, 11, 15\} + \{0, 42\} \\ &= \{0, 1\} + \{0, 10\} + \{0, 4, 10, 11, 15, 19\} + \{0, 42\} \\ &= \{0, 10, 11\} + \{0, 1, 4, 5, 10, 11, 15, 19\} + \{0, 42\}. \end{aligned}$$

X is not relatively cancellative and $L(X) = \{3, 4, 5\}$.

Another application

Let M be an additively written monoid. We say that M is **nontorsion** if $\{na \mid n \in \mathbb{N}\}$ is infinite for some $a \in M$.

Theorem (R., 2025)

Let G be an abelian group, let M be an atomic nontorsion submonoid of G and let $\mathcal{H} \in \{\mathcal{P}_{\text{fin}}(M), \mathcal{P}_{\text{fin},0}(M)\}$.





- (1) The submonoid of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ generated by $\{\{1\}, \{2, n\} \mid n \in \mathbb{N}_{\geq 3}\}$ is contained in $\mathcal{L}(\mathcal{H})$.
- (2) $\{\{k, k+2\} \mid k \in \mathbb{N}_{\geq 2}\} \subseteq \mathcal{L}(\mathcal{H})$.
- (3) \mathcal{H} is fully elastic.

Open problems





Let $\mathcal{H} = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$.

- (1) Is $\mathcal{L}(\mathcal{H}) = \{L(A) \mid A \in \mathcal{H} \text{ is relatively cancellative}\}$? In particular, is there some relatively cancellative $X \in \mathcal{H}$ such that $L(X) = \{2, 3, 4\}$ or $L(X) = \{3, 4, 5\}$
- (2) Are all nonempty finite arithmetical progressions $L \subseteq \mathbb{N}_{\geq 2}$ length sets of \mathcal{H} ?
- (3) Is the conjecture of Fan and Tringali true?





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Thank you for your attention!